

Statistical approximation properties of (p, q) -Szász-Mirakjan Kantorovich operators

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Abstract

The main aim of this study is to introduce statistical approximation properties of (p, q) -Szász Mirakjan Kantorovich operators with the help of the Korovkin type statistical approximation theorem. Rates of statistical convergence by means of the modulus of continuity and the Lipschitz type maximal function are also established.

Keywords and phrases: (p, q) -integers, Statistical convergence, (p, q) -Szász Mirakjan Kantorovich operator, rate of statistical convergence; modulus of continuity; positive linear operators; Korovkin type approximation theorem.

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1 Introduction

Recently, Mursaleen et al [9] applied (p, q) -calculus in approximation theory and introduced the first (p, q) -analogue of Bernstein operators based on (p, q) -integers. Motivated by the work of Mursaleen et al [9], the idea of (p, q) -calculus and its importance. Very recently, Khalid et al. has given a very nice application in computer-aided geometric design and applied these Bernstein basis for construction of (p, q) -Bézier curves and surfaces based on (p, q) -integers which is further generalization of q -Bézier curves and surfaces [15, 23, 24]. For similar works based on (p, q) -integers, one can refer [4, 5, 6, 10, 11, 12, 13, 14, 17, 22, 18, 7].

It was S.N. Bernstein [1] in 1912, who first introduced his famous operators $B_n : C[0, 1] \rightarrow C[0, 1]$ defined for any $n \in \mathbb{N}$ and for any function $f \in C[0, 1]$

$$B_n(f; x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0, 1]. \quad (1.1)$$

and named it Bernstein polynomials to prove the Weierstrass theorem [3]. Later it was found that Bernstein polynomials possess many remarkable properties and has various applications in areas such as approximation theory [3], numerical analysis, computer-aided geometric design, and solutions of differential equations due to its fine properties of approximation [16].

In computer aided geometric design (CAGD), Bernstein polynomials and its variants are used in order to preserve the shape of the curves or surfaces. One of the most important curve in CAGD [21] is the classical Bézier curve [2] constructed with the help of Bernstein basis functions.

The Szasz-Mirakjan operators[25, 26] have an important role in the approximation theory, and their approximation properties have been investigated by many researchers. The Kantorovich type of the Szasz Mirakjan operators was defined as

$$K_n(f; x) = ne^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_{k/n}^{(k+1)/n} f(t) dt, \quad f \in C_\gamma[0, \infty) \quad (1.2)$$

$$\text{where } C_\gamma[0, \infty) = \{f \in C_\gamma[0, \infty) : |f(t)| \leq M(1+t)^\gamma\}$$

In[27], Aral and Gupta defined q-type generalization of Szasz Mirakjan operators as follows:

$$S_n^q(f; q, x) = E_q(-[n]_q \frac{x}{b_n}) \sum_{k=0}^{\infty} f\left(\frac{[k]_q b_n}{[n]_q}\right) \frac{([n]_q x)^k}{[k]_q! (b_n)^k} \quad (1.3)$$

where $f \in C[0, \infty)$, $q \in (0, 1)$, $0 \leq x < \frac{b_n}{1-q_n}$, b_n is a sequence of positive numbers such that $\lim_{n \rightarrow \infty} b_n = \infty$ and $E_q(x) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!}$

Let $C_B[0, \infty)$ be the space of all bounded and continuous functions on $[0, \infty)$. Then $C_B[0, \infty)$ is a normed linear space with $\|f\|_{C_B} = \sup_{x \geq 0} |f(x)|$. Let w be a function of the type of modulus of continuity. The principal properties of the function are the following:

- (i) w is a nonnegative increasing function on $[0, \infty)$,
- (ii) $\lim_{\delta \rightarrow 0} w(\delta) = 0$.

Let H_w be the space of all-real valued functions f defined on $[0, \infty)$ satisfying the following condition:

$$|f(x) - f(y)| \leq w(|x - y|)$$

for any $x, y \geq 0$.

In [28], Gadjiev and Caker proved the Korovkin type theorem which gives the conditions of the convergence of the sequence of linear positive operators to find function

in H_w .

Currently, more useful connections of Korovkin type approximation theory, not only with classical approximation theory, but also other branches of mathematics were given by Altomare and Campiti in [29].

Now we recall the following theorem which was given by gadjiev and Cakar:

Theorem 1.1([28]). Let (A_n) be the sequence of linear positive operators from H_w into $C_B[0, \infty)$ satisfying three conditions

$$\lim_{n \rightarrow \infty} \|A_n(t^\nu; x) - x^\nu\|_{c_B} = 0, \quad \nu = 0, 1, 2.$$

Then for any function $f \in H_w$

$$\lim_{n \rightarrow \infty} \|A_n(f) - f\|_{c_B} = 0$$

Let us give rudiments of

$$(p, q)$$

-calculus.

For each nonnegative integer n , the (p, q) -integer $[n]_{p,q}$ is defined by

$$[n]_{p,q} := \frac{(p^n - q^n)}{(p - q)}, \quad n = 0, 1, 2, \dots, \quad 0 < q < p \leq 1$$

whereas q -integers are given by

$$[n]_q := \frac{(1 - q^n)}{(1 - q)}, \quad n = 0, 1, 2, \dots, \quad 0 < q \leq 1$$

and the (p, q) -binomial coefficients are defined by

$$\left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}! [n-k]_{p,q}!}$$

By some simple calculation, we have the following relation

$$q^k [n - k + 1]_{p,q} = [n + 1]_{p,q} - p^{n-k+1} [k]_{p,q}$$

For details on q -calculus and (p, q) -calculus, one is referred to [20, 30]

Recently in [31], Kantorovich Variant of (p, q) -Szsz-Mirakjan Operators has been studied and defined as follows

For $f \in C[0, \infty)$, $0 < q < p \leq 1$ and each positive integer n

$$K_n(f, p, q; x) = [n]_{p,q} \sum_{k=0}^{\infty} p^{-k} q^k s_{n,k}(p, q; x) \int_{[k]_{p,q}/q^{k-1}[n]_{p,q}}^{[k+1]_{p,q}/q^k[n]_{p,q}} f(t) d_{p,q} t \quad (1.4)$$

where f is a nondecreasing function.

In [31] M. Mursaleen et al. obtained the uniform approximation of these operators to

the function $f \in H_w$ as follows

Theorem 1.2([31]). Let $q = q_n, p = p_n$ be a sequence satisfying $0 < q_n < p_n \leq 1$ and let $q_n \rightarrow 1, p_n \rightarrow 1$ as $n \rightarrow \infty$. If K_n is defined by(1.3), then for any $f \in H_w$

$$\lim_{n \rightarrow \infty} \|K_n(f) - f\|_{c_B} = 0$$

On the other hand the concept of statistical convergence was introduced by Fast [32] in the year 1950 and in recent times it has become an active area of research. The concept of the limit of a sequence has been generalized to a statistical limit through the natural density of a set K of positive integers, defined as

$$\delta(K) = \lim_{n \rightarrow \infty} \frac{1}{n} (k \leq n \text{ for } k \in K)$$

provided this limit exists[33]. We say that the sequence $x = (x_n)$ statistically converges to a number L . if for each $\epsilon > 0$, the density of the set $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$. We denote it by $st - \lim_{k \rightarrow \infty} x_k = L$. It is easily seen that every convergent sequence is statistically convergent but not inversely.

Statistical convergence was used in approximation theory by Gadjiev and Orhan [34]. They proved the statistically Korovkin type theorem for the linear positive operators as follows

Theorem 1.3 ([34]). If the sequence of positive linear operators $A_n : C[a, b] \rightarrow C[a, b]$ satisfies the conditions

$$st - \lim_{n \rightarrow \infty} \|A_n(e_\nu; \cdot) - e_\nu\|_{C[a, b]} = 0$$

with $e_\nu = t^\nu$ for $\nu = 0, 1, 2$

then for any function $f \in C[a, b]$, we have

$$st - \lim_{n \rightarrow \infty} \|A_n(f; \cdot) - f\|_{C[a, b]} = 0$$

The main aim of this paper is to study the operators defined by M.Mursaleen et.al [31] and obtain statistically approximation properties of the operator with the help of the Korovkin type theorem proved by A.D. gadjiev and C. Orhan [34] and estimate the rate of statistically convergence of the sequence of the operator to the function f . we had obtained the ordinary result for the uniform convergence of the bivariate (p, q) -Szász-Mirakjan operators in [31]. In this study, the similar results will be used while proving the theorems.

2 Main Results:

Let us give the following theorem:

Theorem 2.1 : Let (A_n) be the sequence of linear positive operators from H_w into

$C_B(R_+)$ satisfying three conditions

$$st - \lim_{n \rightarrow \infty} \|A_n(t^\nu : x) - x^\nu\|_{c_B} = 0, \quad \nu = 0, 1, 2$$

then for any $f \in H_w$

$$st - \lim_{n \rightarrow \infty} \|A_n(f; \cdot) - f\|_{c_B} = 0$$

Notice that this theorem is one variable case of the Duman and Erkus theorem [35]
To obtain the statistically convergence of the operators above, we need the following three lemmas as given in [31]

Lemma 2.2 For $x \geq 0$, $0 < q < p \leq 1$

$$(i) \quad K_n(1, p, q; x) = 1$$

$$(ii) \quad K_n(t, p, q; x) = \frac{1}{q}x + \frac{1}{[2]_{p,q}[n]_{p,q}}$$

$$(iii) \quad K_n(t^2, p, q; x) = \frac{p}{q^3}x^2 + \left(\frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} + \frac{1}{q^2[n]_{p,q}}\right)x + \frac{1}{[3]_{p,q}[n]_{p,q}^2}$$

3 Korovkin type statistical approximation properties

The main aim of this paper is to obtain the Korovkin type statistical approximation properties of operators defined in(1.3), with the help of Theorem (1.1).

Now, we consider a sequence $p = p_n$, $q = q_n$ satisfying the following expression

$$st - \lim_n q_n = 1, \quad st - \lim_n p_n = 1, \quad 0 < q_n < p_n \leq 1 \quad (3.1)$$

Theorem 3.1 Let (K_n) be the sequence of the operators (1.3) and the sequence $q = q_n$, $p = p_n$ satisfies (1.3) for $0 < q_n < p_n \leq 1$ then for any $f \in H_w$

$$st - \lim_{n \rightarrow \infty} \|K_n(f; q_n; p_n; \cdot) - f\|_{c_B} = 0$$

Proof: In the light of Theorem 2.1, it is sufficient to prove the followings:

$$st - \lim_{n \rightarrow \infty} \|K_n(t^\nu : x) - x^\nu\|_{c_B} = 0, \quad \nu = 0, 1, 2$$

From Lemma , the first condition of above equation is easily obtained for $\nu = 0$.
for $\nu = 1$

$$\|K_n(t; q_n; p_n; x) - x\|_{c_B} = \left\| \frac{1}{q_n}x + \frac{1}{[2]_{p_n, q_n}[n]_{p_n, q_n}} - x \right\| \leq \left| \left(\frac{1}{q_n} - 1 \right) + \frac{1}{[2]_{p_n, q_n}[n]_{p_n, q_n}} \right|$$

Now for a given $\epsilon > 0$, we define following sets

$$U = \{n : \|K_n(t; q_n; p_n; x) - x\| \geq \epsilon\}$$

$$U_1 = \{n : 1 - \frac{1}{q_n} \geq \epsilon\}, \quad U_2 = \{n : \frac{1}{[2]_{p_n, q_n} [n]_{p_n, q_n}} \geq \epsilon\}$$

It is obvious that $U \subset U_1 \cup U_2$ Therefore it can be written as

$$\delta\{k \leq n : \|K_n(t; q_n; p_n; x) - x\| \geq \epsilon\} \leq \delta\{k \leq n : 1 - \frac{1}{q_n} \geq \epsilon\} + \delta\{k \leq n : \frac{1}{[2]_{p_n, q_n} [n]_{p_n, q_n}} \geq \epsilon\}$$

By using (3.1), it is clear that

$$\begin{aligned} st - \lim_{n \rightarrow \infty} (1 - \frac{1}{q_n}) &= 0 \text{ and } st - \lim_{n \rightarrow \infty} (\frac{1}{[2]_{p_n, q_n} [n]_{p_n, q_n}}) = 0 \\ \text{so } \delta\{k \leq n : 1 - \frac{1}{q_n} \geq \epsilon\} &= 0 \text{ and } \delta\{k \leq n : \frac{1}{[2]_{p_n, q_n} [n]_{p_n, q_n}} \geq \epsilon\} = 0 \\ st - \lim_{n \rightarrow \infty} \|K_n(t; q_n; p_n; \cdot) - x\|_{c_B} &= 0 \end{aligned}$$

Lastly for $\nu = 2$ we have

$$\begin{aligned} \|K_n(t^2; q_n; p_n; x) - x^2\| &= \sup_{x \geq 0} \left\{ x^2 \left(\frac{p_n}{q_n^3} - 1 \right) + x \left(\frac{p_n + [2]_{p_n, q_n}}{q_n [3]_{p_n, q_n} [n]_{p_n, q_n}} + \frac{1}{q_n^2 [n]_{p_n, q_n}} \right) + \frac{1}{[3]_{p_n, q_n} [n]_{p_n, q_n}^2} \right\} \\ &\leq \left| \frac{p_n}{q_n^3} - 1 \right| + \left| \frac{p_n + [2]_{p_n, q_n}}{q_n [3]_{p_n, q_n} [n]_{p_n, q_n}} + \frac{1}{q_n^2 [n]_{p_n, q_n}} \right| + \left| \frac{1}{[3]_{p_n, q_n} [n]_{p_n, q_n}^2} \right| \\ &= \frac{p_n}{q_n^3} - 1 + \frac{p_n + [2]_{p_n, q_n}}{q_n [3]_{p_n, q_n} [n]_{p_n, q_n}} + \frac{1}{q_n^2 [n]_{p_n, q_n}} + \frac{1}{[3]_{p_n, q_n} [n]_{p_n, q_n}^2} \end{aligned}$$

$$\text{If we choose } \alpha_n = \frac{p_n}{q_n^3} - 1, \beta_n = \frac{p_n + [2]_{p_n, q_n}}{q_n [3]_{p_n, q_n} [n]_{p_n, q_n}} + \frac{1}{q_n^2 [n]_{p_n, q_n}}, \gamma_n = \frac{1}{[3]_{p_n, q_n} [n]_{p_n, q_n}^2}$$

then one can write

$$st - \lim_{n \rightarrow \infty} \alpha_n = st - \lim_{n \rightarrow \infty} \beta_n = st - \lim_{n \rightarrow \infty} \gamma_n = 0 \quad (3.2)$$

by (3.1) Now given $\epsilon > 0$, we define the following four sets

$$\begin{aligned} U &= \{n : \|K_n(t^2; q_n; p_n; x) - x^2\| \geq \epsilon\}, \\ U_1 &= \{n : \alpha_n \geq \frac{\epsilon}{3}\}, \quad U_2 = \{n : \beta_n \geq \frac{\epsilon}{3}\} \quad U_3 = \{n : \gamma_n \geq \frac{\epsilon}{3}\} \end{aligned}$$

It is obvious that $U \subseteq U_1 \cup U_2 \cup U_3$. Then we obtain

$$\begin{aligned} \delta\{k \leq n : \|K_n(t^2; q_n; p_n; x) - x^2\| \geq \epsilon\} &\leq \delta\{k \leq n : \alpha_n \geq \frac{\epsilon}{3}\} + \delta\{k \leq n : \beta_n \geq \frac{\epsilon}{3}\} \\ &\quad + \delta\{k \leq n : \gamma_n \geq \frac{\epsilon}{3}\} \end{aligned}$$

So the right hand side of the inequalities is zero by (3.2), then

$$st - \lim_{n \rightarrow \infty} \|K_n(t^2; q_n; p_n; \cdot) - x^2\|_{c_B} = 0$$

holds. Hence the proof follows from theorem (2.1).

4 Rates of statistical convergence

In this section, we give the rates of statistical convergence of the operator (1.3) by means of modulus of continuity and Lipschitz type maximal functions. The modulus of continuity for the functions $f \in H_w$ is defined as

$$w(f; \delta) = \sup_{x, t \geq 0, |t-x| < \delta} |f(t) - f(x)|$$

where $w(f; \delta)$ for $\delta > 0$ satisfies the following conditions: for every $f \in H_w$

$$(i) \lim_{\delta \rightarrow 0} w(f; \delta) = 0$$

$$(ii) |f(t) - f(x)| \leq w(f; \delta) \left(\frac{|t-x|}{\delta} + 1 \right) \quad (4.1)$$

Theorem 4.1. Let the sequence $q = (q_n)$, $p = (p_n)$ satisfies the condition in (3.1) and $0 < q_n < p_n \leq 1$. Then we have

$$|K_n(f; q_n; p_n; x) - f(x)| \leq 2w(f; \sqrt{\delta_n(x)})$$

where

$$\delta_n(x) = x^2 \left(\frac{p_n}{q_n^3} - \frac{2}{q_n} + 1 \right) + x \left(\frac{p_n + [2]_{p_n, q_n}}{q_n [3]_{p_n, q_n} [n]_{p_n, q_n}} + \frac{1}{q_n^2 [n]_{p_n, q_n}} - \frac{2}{[2]_{p_n, q_n} [n]_{p_n, q_n}} \right) + \frac{1}{[3]_{p_n, q_n} [n]_{p_n, q_n}^2} \quad (4.2)$$

Proof.

Since $|K_n(f; q_n; p_n; x) - f(x)| \leq K_n(|f(t) - f(x)|; q_n; p_n; x)$, by (4.1) we get

$$|K_n(f; q_n; p_n; x) - f(x)| \leq w(f; \delta) \left(\{K_n(1; q_n; p_n; x) + \frac{1}{\delta_n} K_n(|t - x|; q_n; p_n; x)\} \right).$$

Using Cauchy-Schwartz inequality, we have

$$\begin{aligned} |K_n(f; q_n; p_n; x) - f(x)| &\leq w(f; \delta_n) \left(1 + \frac{1}{\delta_n} [(K_n(|t - x|^2; q_n; p_n; x))]^{\frac{1}{2}} [K_n(1; q_n; p_n; x)]^{\frac{1}{2}} \right) \\ &\leq w(f; \delta_n) \left\{ 1 + \frac{x^2}{\delta_n} \left(\frac{p_n}{q_n^3} - \frac{2}{q_n} + 1 \right) \right. \\ &\quad + \frac{x}{\delta_n} \left(\frac{p_n + [2]_{p_n, q_n}}{q_n [3]_{p_n, q_n} [n]_{p_n, q_n}} + \frac{1}{q_n^2 [n]_{p_n, q_n}} - \frac{2}{[2]_{p_n, q_n} [n]_{p_n, q_n}} \right) \\ &\quad \left. + \frac{1}{\delta_n} \left(\frac{1}{[3]_{p_n, q_n} [n]_{p_n, q_n}^2} \right) \right\} \end{aligned}$$

By choosing δ_n as in (4.2), we get the desired result.

This completes the proof of the theorem.

Note that in condition (3.1),

$$st - \lim_{n \rightarrow \infty} \delta_n = 0.$$

By (4.1) we have

$$st - \lim_{n \rightarrow \infty} w(f; \delta_n) = 0,$$

which gives us the pointwise rate of statistical convergence of the operator $K_n(f; q_n; p_n x)$ to $f(x)$.

Now we will give an estimate concerning the rate of approximation by means of Lipschitz type maximal functions. In [36], B. Lenze introduced a Lipschitz type maximal function as

$$\tilde{f}_\alpha(x) = \sup_{t > 0, t \neq x} \frac{|f(t) - f(x)|}{|x - t|^\alpha}.$$

In [37], the Lipschitz type maximal function space on $E \subset [0, \infty)$ is defined as follows

$$\tilde{W}_\alpha = \{f = \sup(1+x)^\alpha \tilde{f}_\alpha(x) \leq M \frac{1}{(1+y)^\alpha}; x \geq 0 \text{ and } y \in E\},$$

where f is bounded and continuous function on $[0, \infty)$, M is a positive constant and $0 < \alpha \leq 1$.

we denote by $d(x, E)$, the distance between x and E , that is $d(x, E) = \inf\{|x - y|; y \in E\}$

Theorem 4.2. If K_n be defined by (1.3), then for all $f \in \tilde{W}_{\alpha, E}$

$$|K_n(f; q_n; p_n; x) - f(x)| \leq M \left(\delta_n^{\frac{\alpha}{2}} + 2(d(x, E))^\alpha \right) \quad (4.3)$$

where $\delta_n(x)$ is defined in Theorem (4.1)

Proof. Let $x \geq 0$, $(x, x_0) \in [0, \infty) \times E$. Then we have

$$|f - f(x)| \leq |f - f(x_0)| + |f(x_0) - f(x)|.$$

Since K_n is a positive and linear operator, $f \in \tilde{W}_{\alpha, E}$ and using the above inequality

$$\begin{aligned} |K_n(f; q_n; p_n; x) - f(x)| &\leq |K_n(|f - f(x_0)|; q_n; p_n; x) - f(x)| + |f(x_0) - f(x)| K_n(1; q_n; p_n; x) \\ &\leq M(K_n(|t - x_0|^\alpha; q_n; p_n; x) + |x - x_0|^\alpha K_n(1; q_n; p_n; x)) \end{aligned}$$

Therefore we have

$$K_n(|t - x_0|^\alpha; q_n; p_n; x) \leq K_n(|t - x|^\alpha; q_n; p_n; x) + |x - x_0|^\alpha K_n(1; q_n; p_n; x).$$

By using the Holder inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have

$$\begin{aligned} K_n(|t - x|^\alpha; q_n; p_n; x) &\leq K_n((t - x)^2; q_n; p_n; x)^{\frac{\alpha}{2}} (K_n(1; q_n; p_n; x))^{\frac{2-\alpha}{2}} + |x - x_0|^\alpha K_n(1; q_n; p_n; x) \\ &= \delta_n^{\frac{\alpha}{2}} + |x - x_0|^\alpha. \end{aligned}$$

This completes the proof of the theorem.

Remark 4.1 If we take $E = [0, \infty)$ in Theorem (4.2), since $d(x, E) = 0$, then we obtain the following result:

For every $f \in \tilde{W}_{\alpha, [0, \infty)}$

$$| K_n(f; q_n; p_n x) - f(x) | \leq M \delta_n^{\frac{\alpha}{2}}.$$

where δ_n is defined as in (12).

Remark 4.2 By using (3.1), It is easy to verify that

$$\lim_{n \rightarrow \infty} \delta_n = 0.$$

That is, the rate of statistical convergence of (1.3) are estimated by means of Lipschitz type maximal functions.

5 Construction of the bivariate operators

In what follows we construct the bivariate extension of the operators (1.3). We will introduce the statistical convergence of the operators to a function f and investigate the statistical rate of convergence of these operators.

Let $R_+^2 = [0, \infty) \times [0, \infty)$, $f : R_+^2 \rightarrow R$ and $0 < p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2} \leq 1$ Then we define the bivariate companion of the operators (1.3) as follows:

$$\begin{aligned} & K_{n_1, n_2}(f, p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) = \\ & = [n_1]_{p_{n_1}, q_{n_1}} [n_2]_{p_{n_2}, q_{n_2}} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} p_{n_1}^{-k_{n_1}} q_{n_1}^{k_{n_1}} s_{n_1, k_1}(p_{n_1}, q_{n_1}; x, y) p_{n_2}^{-k_{n_2}} q_{n_2}^{k_{n_2}} s_{n_2, k_2}(p_{n_2}, q_{n_2}; x, y) \\ & \times \int_{[k_1]_{p_{n_1}, q_{n_1}} / q_{n_1}^{k_1-1} [n_1]_{p_{n_1}, q_{n_1}}}^{[k_1+1]_{p_{n_1}, q_{n_1}} / q_{n_1}^{k_1} [n_1]_{p_{n_1}, q_{n_1}}} \int_{[k_2]_{p_{n_2}, q_{n_2}} / q_{n_2}^{k_2-1} [n_2]_{p_{n_2}, q_{n_2}}}^{[k_2+1]_{p_{n_2}, q_{n_2}} / q_{n_2}^{k_2} [n_2]_{p_{n_2}, q_{n_2}}} f(t) d_{p_{n_1}, q_{n_1}} t d_{p_{n_2}, q_{n_2}} t \quad (5.1) \end{aligned}$$

For $K = [0, \infty)[0, \infty)$, the modulus of continuity for the bivariate case is defined as

$$w_2(f; \delta_1, \delta_2) = \sup | f(u, v) - f(x, y) | : (u, v), (x, y) \in K \text{ and } | u - x | \leq \delta_1, | v - y | \leq \delta_2$$

and $w_2(f; \delta_1, \delta_2)$ satisfy the following condition

$$| f(u, v) - f(x, y) | \leq w_2(f; | u - x |, | v - y |)$$

for each $f \in H_{w_2}$. Detailed study of modulus of continuity for the bivariate analogue one is referred to [38].

The first Korovkin type theorem for the statistical approximation for the bivariate

analogue of linear positive operators defined in the space H_{w_2} was obtained by Erkus and Duman [35] which is as follows

Theorem 5.1[35]. Let K_n be a sequence of positive linear operators from H_{w_2} into $C_B(K)$. Then for each $f \in H_{w_2}$,

$$st - \lim_{n \rightarrow \infty} \|K_n(f) - f\| = 0$$

is satisfied if the following holds:

$$st - \lim_{n \rightarrow \infty} \|K_n(f_i) - f\| = 0, \quad i = 0, 1, 2, 3$$

where

$$f_0(u, v) = 1, \quad f_1(u, v) = u, \quad f_2(u, v) = v, \quad f_3(u, v) = u^2 + v^2 \quad (5.2)$$

To study the statistical convergence of the bivariate operators, the following

Lemma 5.2. The bivariate operators defined above satisfy the followings:

- (i) $K_{n_1, n_2}(f_0, p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) = 1$
- (ii) $K_{n_1, n_2}(f_1, p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) = \frac{1}{q_{n_1}}x + \frac{1}{[2]_{p_{n_1}, q_{n_1}}[n_1]_{p_{n_1}, q_{n_1}}}$
- (iii) $K_{n_1, n_2}(f_2, p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) = \frac{1}{q_{n_2}}y + \frac{1}{[2]_{p_{n_2}, q_{n_2}}[n_2]_{p_{n_2}, q_{n_2}}}$
- (iv) $K_{n_1, n_2}(f_3, p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) = x^2\left(\frac{p_{n_1}}{q_{n_1}^3} - 1\right) + x\left(\frac{p_{n_1} + [2]_{p_{n_1}, q_{n_1}}}{q_{n_1}[3]_{p_{n_1}, q_{n_1}}[n_1]_{p_{n_1}, q_{n_1}}} + \frac{1}{q_{n_1}^2[n_1]_{p_{n_1}, q_{n_1}}}\right) + \frac{1}{[3]_{p_{n_1}, q_{n_1}}[n_1]_{p_{n_1}, q_{n_1}}^2} + y^2\left(\frac{p_{n_2}}{q_{n_2}^3} - 1\right) + y\left(\frac{p_{n_2} + [2]_{p_{n_2}, q_{n_2}}}{q_{n_2}[3]_{p_{n_2}, q_{n_2}}[n_2]_{p_{n_2}, q_{n_2}}} + \frac{1}{q_{n_2}^2[n_2]_{p_{n_2}, q_{n_2}}}\right) + \frac{1}{[3]_{p_{n_2}, q_{n_2}}[n_2]_{p_{n_2}, q_{n_2}}^2}$

Proof. Exploiting the proofs for the bivariate operators in [39], the above can be easily established. So we skip the proof.

Now, we consider a sequence $p = p_{n_1}$, $p = p_{n_2}$, $q = q_{n_1}$, $q = q_{n_2}$, be statistically convergent to unity but not convergent in usual sense, so we can write them for

$$0 < p_{n_1}, p_{n_2}, q_{n_1}, q_{n_2} \leq 1$$

$$st - \lim_n q_{n_1} = st - \lim_n p_{n_1} = st - \lim_n q_{n_2} = st - \lim_n p_{n_2} = 1 \quad (5.3)$$

Now under the condition in (5.3), let us show that the statistical convergence of the bivariate operator (5.1) with the help of the proof of Theorem (3.1).

Theorem 5.3. Let $p = (p_{n_1})$, $p = (p_{n_2})$, $q = (q_{n_1})$ and $q = (q_{n_2})$ be the sequences satisfy the conditions (4.3) and let K_{n_1}, K_{n_2} be the sequence of linear positive operators from $H_{w_2}(R_+^2)$ into $C_B(R_+)$. Then for each $f \in H_{w_2}$,

$$st - \lim_{n_1, n_2 \rightarrow \infty} \|K_{n_1, n_2}(f) - f\| = 0$$

Proof. With the aid of the Lemma (5.2), a proof similar to the proof of the Theorem (3.1) can be easily obtained. So we shall omit the proof.

6 Rates of convergence of the bivariate operators

For any $f \in H_{w_2}(R_+^2)$, the modulus of continuity of the bivariate analogue is defined as:

$$\tilde{w}(f; \delta_1, \delta_2) = \sup_{t, x \geq 0} \{ |f(t, s) - f(x, y)| : |t - x| \leq \delta_1, |s - y| \leq \delta_2, (t, s), (x, y) \in R_+^2 \}$$

For details of this sort of modulus, one is referred to [38]. Here $\tilde{w}(f; \delta_1, \delta_2)$ satisfies the following conditions

$$\begin{aligned} (i) \quad & \tilde{w}(f; \delta_1, \delta_2) \rightarrow 0 \text{ if } \delta_1 \rightarrow 0, \delta_2 \rightarrow 0 \\ (ii) \quad & |f(t, s) - f(x, y)| \leq \tilde{w}(f; \delta_1, \delta_2) \left(\frac{|t-x|}{\delta_1} + 1 \right) \left(\frac{|s-y|}{\delta_2} + 1 \right) \end{aligned} \quad (6.1)$$

Now in the following theorem we study the rate of statistical convergence of the bivariate operators through modulus of continuity in H_{w_2} .

Theorem 6.1. Let $p = (p_{n_1})$, $p = (p_{n_2})$, $q = (q_{n_1})$, $q = (q_{n_2})$ be four sequences obeying conditions of (4.3). Then we have

$$|K_{n_1, n_2}(f, p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) - f(x, y)| \leq 4w\left(f; \sqrt{\delta_{n_1}(x)}; \sqrt{\delta_{n_2}(y)}\right)$$

where

$$\begin{aligned} \delta_{n_1}(x) &= x^2 \left(\frac{p_{n_1}}{q_{n_1}^3} - \frac{2}{q_{n_1}} + 1 \right) + x \left(\frac{p_{n_1} + [2]_{p_{n_1}, q_{n_1}}}{q_{n_1} [3]_{p_{n_1}, q_{n_1}} [n_1]_{p_{n_1}, q_{n_1}}} + \frac{1}{q_{n_1}^2 [n_1]_{p_{n_1}, q_{n_1}}} - \frac{2}{[2]_{p_{n_1}, q_{n_1}} [n_1]_{p_{n_1}, q_{n_1}}} \right) \\ &\quad + \frac{1}{[3]_{p_{n_1}, q_{n_1}} [n_1]_{p_{n_1}, q_{n_1}}^2} \\ \delta_{n_2}(y) &= y^2 \left(\frac{p_{n_2}}{q_{n_2}^3} - \frac{2}{q_{n_2}} + 1 \right) + y \left(\frac{p_{n_2} + [2]_{p_{n_2}, q_{n_2}}}{q_{n_2} [3]_{p_{n_2}, q_{n_2}} [n_2]_{p_{n_2}, q_{n_2}}} + \frac{1}{q_{n_2}^2 [n_2]_{p_{n_2}, q_{n_2}}} - \frac{2}{[2]_{p_{n_2}, q_{n_2}} [n_2]_{p_{n_2}, q_{n_2}}} \right) \\ &\quad + \frac{1}{[3]_{p_{n_2}, q_{n_2}} [n_2]_{p_{n_2}, q_{n_2}}^2} \end{aligned}$$

Proof: Using the property of the modulus above, we have

$$\begin{aligned} |K_{n_1, n_2}(f, p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) - f(x, y)| &\leq w(f; \delta_{n_1}, \delta_{n_2}) \left(\{K_{n_1, n_2}(f_0; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) \right. \\ &\quad \left. + \frac{1}{\delta_{n_1}} K_{n_1, n_2}(|t - x|; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) \} \{K_{n_1, n_2}(f_0; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) \right. \\ &\quad \left. + \frac{1}{\delta_{n_2}} K_{n_1, n_2}(|s - y|; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) \} \right) \end{aligned}$$

Applying the Cauchy-Schwartz inequality, we get

$$\begin{aligned} K_{n_1, n_2}(|t - x|; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) &\leq [(K_{n_1, n_2}(|t - x|^2; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y))]^{\frac{1}{2}} \\ &\sim \times [K_{n_1, n_2}(f_0; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y)]^{\frac{1}{2}} \end{aligned}$$

On substituting this in the above inequality, we get the proof of the theorem.

Now we shall study the statistical convergence of the bivariate operators using Lipschitz type maximal functions.

The Lipschitz type maximal function space on $E \times E \subset R_+ \times R_+$ is defined as follows

$$\tilde{W}_{\alpha_1, \alpha_2} E^2 = \{f : \sup(1+t)^{\alpha_1}(1+s)^{\alpha_2} \tilde{f}_{\alpha_1, \alpha_2}(x, y) \leq M \frac{1}{(1+x)^{\alpha_1}} \frac{1}{(1+y)^{\alpha_2}}; x, y \geq 0, (t, s) \in E^2\} \quad (6.2)$$

Where f is a bounded and continuous function on R_+ , M is a positive constant and $0 \leq \alpha_1, \alpha_2 \leq 1$ and $\tilde{f}_{\alpha_1, \alpha_2}(x, y)$ is defined as follows:

$$\tilde{f}_{\alpha_1, \alpha_2}(x, y) = \sup_{t, s \geq 0} \frac{|f(t, s) - f(x, y)|}{|t - x|^{\alpha_1} |s - y|^{\alpha_2}}.$$

Theorem 6.2. Let $p = (p_{n_1})$, $p = (p_{n_2})$, $q = (q_{n_1})$, $q = (q_{n_2})$ be four sequences satisfying the conditions of (5.2). Then we have

$$\begin{aligned} |K_{n_1, n_2}(f; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) - f(x, y)| &\leq M_{p_{n_1}, q_{n_1}, p_{n_2}, q_{n_2}} \left(\delta_{n_1}(x)^{\frac{\alpha_1}{2}} \delta_{n_2}(y)^{\frac{\alpha_2}{2}} (p_{n_1}, q_{n_1}, p_{n_2}, q_{n_2}) \right. \\ &\quad \left. + \delta_{n_1}(x)^{\frac{\alpha_1}{2}} d(y, E)^{\alpha_2} + \delta_{n_2}(y)^{\frac{\alpha_2}{2}} d(x, E)^{\alpha_1} + 2 d(x, E)^{\alpha_1} d(y, E)^{\alpha_2} \right) \end{aligned}$$

where

$0 \leq \alpha_1, \alpha_2 \leq 1$ and $\delta_{n_1}, \delta_{n_2}$ are defined as in Theorem (6.1) and

$d(x, E) = \inf\{|x - y| : y \in E\}$.

Proof. For $x, y \geq 0$ and $(x_1, y_1) \in E \times E$, we can write

$$|f(t, s) - f(x, y)| \leq |f(t, s) - f(x_0, y_0)| + |f(x_0, y_0) - f(x, y)|.$$

Applying the operator K_{n_1, n_2} to both sides of the above inequality and making use of eqn (6.2), we have

$$\begin{aligned} |K_{n_1, n_2}(f; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) - f(x, y)| &\leq |K_{n_1, n_2}(|f(t, s) - f(x_0, y_0)|; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) \\ &\quad + |K_{n_1, n_2}(f(x_0, y_0) - f(x, y)); p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y)| \\ &\leq MK_{n_1, n_2}(|t - x_0|^{\alpha_1} |s - y_0|^{\alpha_2}; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) \\ &\quad + M |x - x_0|^{\alpha_1} |y - y_0|^{\alpha_2} K_{n_1, n_2}(f_0; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) \end{aligned} \quad (6.3)$$

Now for $0 \leq p \leq 1$, using $(a + b)^p \leq a^p + b^p$, we can write

$$|t - x_0|^{\alpha_1} \leq |t - x|^{\alpha_1} + |x - x_0|^{\alpha_1}$$

and

$$|s - y_0|^{\alpha_2} \leq |s - y|^{\alpha_2} + |y - y_0|^{\alpha_2}$$

Using these inequalities in(6.3), we get

$$\begin{aligned}
| K_{n_1, n_2}(f; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) - f(x, y) | &\leq K_{n_1, n_2}(| t - x |^{\alpha_1} | s - y |^{\alpha_2}; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) \\
&+ | y - y_0 |^{\alpha_2} K_{n_1, n_2}(| t - x |^{\alpha_1}; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) \\
&+ | x - x_0 |^{\alpha_1} K_{n_1, n_2}(| s - y |^{\alpha_2}; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) \\
&+ | x - x_0 |^{\alpha_1} | y - y_0 |^{\alpha_2} K_{n_1, n_2}(f_0; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y)
\end{aligned}$$

Now using the Holders inequality for $p_1 = \frac{2}{\alpha_1}$, $p_2 = \frac{2}{\alpha_2}$ $q_1 = \frac{2}{2-\alpha_1}$ $q_2 = \frac{2}{2-\alpha_2}$, we get

$$\begin{aligned}
&K_{n_1, n_2}(| t - x |^{\alpha_1} | s - y |^{\alpha_2}; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) \\
&= K_{n_1, n_2}(| t - x |^{\alpha_1}; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) K_{n_1, n_2}(| s - y |^{\alpha_2}; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) \\
&\leq [(K_{n_1, n_2}(| t - x |^2; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y)]^{\frac{\alpha_1}{2}} [K_{n_1, n_2}(f_0; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y)]^{\frac{2-\alpha_1}{2}} \\
&\times [(K_{n_1, n_2}(| s - y |^2; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y)]^{\frac{\alpha_2}{2}} [K_{n_1, n_2}(f_0; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y)]^{\frac{2-\alpha_2}{2}}
\end{aligned}$$

This consequently gives the expected result. so the proof is complete. \square

Remark 6.3. If we take $E = [0, \infty)$, then because of $d(x, E) = 0$ and $d(y, E) = 0$, we have

$$| K_{n_1, n_2}(f; p_{n_1}, q_{n_1}; p_{n_2}, q_{n_2}; x, y) - f(x, y) | \leq M(p_{n_1}, q_{n_1}, p_{n_2}, q_{n_2})^{4 - \frac{\alpha_1 + \alpha_2}{2}} \delta_{n_1}(x)^{\frac{\alpha_1}{2}} \delta_{n_2}(y)^{\frac{\alpha_2}{2}}$$

Remark 6.4. Using (5.3), it can be easily verified that $st - \lim_{n_1} \delta_{n_1} = 0$ and $st - \lim_{n_2} \delta_{n_2} = 0$. So we can estimate the order of statistical approximation of our bivariate operators by means of Lipschitz type maximal functions using this result.

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